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# Some Recent Results on Isospectral Flows

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## Abstract

The following two results related to recent studies on isospectral flows are presented. (i) The multiple bracket generalizations of Brockett's double bracket equation  $\dot{H} = [H, [H, N]]$  are proved to have similar properties to Brockett's equation. (ii) The general formula of isospectral gradient flows, through which we can obtain useful examples, is derived. A new dynamical system which provides a new computational algorithm for solving the eigenvalue problem of non-symmetric matrices is calculated as an example.

## 1 Introduction

Matrix dynamical systems whose solutions evolve with their eigenvalues preserved are called "isospectral flows". Recently, isospectral flows have been extensively studied because they are related to the theory of solitons and the eigenvalue problem of matrices. The Toda equation[3][4] which is fundamental in the theory of solitons can be represented as an isospectral flow on the set of tridiagonal matrices [5]. Further, it has been proved that the Toda flow converges to a diagonal matrix whose diagonal entries are sorted with regard to their absolute values. P.Deift et al. suggested the use of the Toda flow as a computational algorithm for solving the eigenvalue problem of symmetric matrices[6].

Brockett[1] introduced the dynamical system on the set of real  $n \times n$  matrices which is expressed in the double bracket form

$$\dot{H}(t) = [H(t), [H(t), N]] \quad (1)$$

where  $[A, B] = AB - BA$ ,  $N$  is a fixed real diagonal matrix with distinct diagonal entries, and  $H(0)$  is a real symmetric matrix. He proved that (i) the dynamical

system is an isospectral flow on the set of real symmetric matrices and (ii) the solution of the system converges to a diagonal matrix and the diagonal entries of  $H(\infty)$  and  $N$  are similarly ordered. He also showed that we can sort lists, diagonalize matrices, and solve linear programming problems using this dynamical system. It has been shown that Brockett's dynamical system includes the Toda equation and the Riccati equation as special cases[7].

There are two main purposes of this paper. One is to generalize Brockett's double bracket equation to a multiple bracket equation

$$\dot{H} = \underbrace{[H, [H, \dots, [H, N] \dots]]}_{m\text{-fold}}$$

and show that this generalized equation has properties similar to Brockett's equation when the multiplicity  $m$  is an even number. The other is to derive the general formula of isospectral gradient flows and obtain useful examples of isospectral flows using the formula.

This paper is organized as follows. In section 2, we generalize Brockett's double bracket equation to the multiple bracket equations. In section 3, we derive the general formula of isospectral gradient flows and present some useful examples obtained through the use of this formula. The final section contains concluding remarks.

## 2 Multiple Bracket Equations

### 2.1 Simple generalization

In this section, we investigate the dynamical systems of the multiple bracket form

$$\dot{H}(t) = \underbrace{[H(t), [H(t), \dots, [H(t), N] \dots]]}_{m\text{-fold}}, \quad (2)$$

where  $H(t)$  is a real  $n \times n$  matrix and  $N$  is a fixed real  $n \times n$  matrix.

Multiple bracket equations are classified into two families according to the convergence properties of their solutions, that is, into the families of even and odd multiplicities. The equations whose multiplicity  $m$  is even have very similar properties to Brockett's double bracket equation, but the equations whose multiplicity  $m$  is odd have completely different convergence properties from Brockett's equation. The properties of the solutions of (2) in the case that  $N$  is a diagonal matrix with distinct diagonal entries and  $H(0)$  is symmetric are summarized as follows.

- If  $m$  is even, (2) defines an isospectral flow on the set of real symmetric matrices and the solution of (2) converges to the diagonal matrix whose diagonal entries are similarly ordered to the diagonal entries of  $N$ .
- If  $m$  is odd, (2) defines an isospectral flow but has no asymptotically stable fixed point.

The rest of this section constructs the proof for the above mentioned properties.

**Theorem 2.1** *Suppose that  $N$  and  $H(0)$  are symmetric and  $m$  is even. Then the ordinary differential equation (2) defines an isospectral flow on the set of real symmetric matrices. The solution  $H(t)$  of (2) exists for all  $t \in R$ .*

**Proof.** Let  $Sym(n, R)$  and  $Skew(n, R)$  denote the set of all real  $n \times n$  symmetric and skew-symmetric matrices respectively,

$$\begin{aligned} Sym(n, R) &= \{X \in M(n, R) \mid X^t = X\}, \\ Skew(n, R) &= \{X \in M(n, R) \mid X^t = -X\}. \end{aligned}$$

By using the following two obvious facts

$$\begin{aligned} X \in Sym(n, R), Y \in Sym(n, R) &\implies [X, Y] \in Skew(n, R), \\ X \in Sym(n, R), Y \in Skew(n, R) &\implies [X, Y] \in Sym(n, R), \end{aligned}$$

we get

$$\underbrace{[H(t), [H(t), \dots, [H(t), N] \dots]]}_{m\text{-fold}} \in Sym(n, R) \quad (m : \text{even}), \quad (3)$$

$$\underbrace{[H(t), [H(t), \dots, [H(t), N] \dots]]}_{m\text{-fold}} \in Skew(n, R) \quad (m : \text{odd}), \quad (4)$$

where  $H(t)$  and  $N$  are symmetric. From (3), it is immediately obtained that the solution  $H(t)$  of (2) evolves on the set of real symmetric matrices where  $m$  is even. Let the  $n \times n$  matrix  $\Theta(t)$  be the solution of the ordinary differential equation

$$\dot{\Theta}(t) = \Theta(t)A(t), \quad \Theta(0) = I_n,$$

where

$$A(t) = \underbrace{[H(t), [H(t), \dots, [H(t), N] \dots]]}_{(m-1)\text{-fold}}. \quad (5)$$

We can readily verify that

$$H(t) = \Theta(t)^{-1}H(0)\Theta(t)$$

satisfies (2) and hence  $H(t)$  is isospectral to  $H(0)$  for all  $t \in R$ . Further, from (4), we see that  $A(t)$  defined by (5) is skew-symmetric when  $m$  is even and thus  $\Theta(t)$  evolves on  $SO(n)$ . Since  $SO(n)$  is compact,  $\Theta(t)$  exists for all  $t \in R$  and thus  $H(t)$  exists for all  $t \in R$ .  $\square$

**Lemma 2.1** *Suppose that  $N$  is a diagonal matrix with distinct diagonal entries and  $H$  is diagonalizable. Then the following two statements are equivalent for  $m = 1, 2, 3, \dots$ .*

- (a)  $\underbrace{[H, [H, \dots, [H, N] \dots]]}_{m\text{-fold}} = 0.$
- (b)  $H$  is a diagonal matrix.

**Proof.** (b) $\Rightarrow$ (a) is obvious. We shall prove (a) $\Rightarrow$ (b).

Suppose that  $H$  is a diagonalizable matrix with eigenvalues  $\lambda_1, \dots, \lambda_n$  occurring with multiplicities  $n_1, \dots, n_r$  ( $\sum_{i=1}^r n_i = n$ ), that is,

$$\lambda_1 = \dots = \lambda_{n_1}, \lambda_{n_1+1} = \dots = \lambda_{n_1+n_2}, \dots, \lambda_{n_1+\dots+n_{r-1}+1} = \dots = \lambda_n.$$

Further, let us assume that  $P \in GL(n, C)$  diagonalizes  $H$  so that,

$$\begin{aligned} P^{-1}HP &= \text{diag}(\lambda_1, \dots, \lambda_n) \\ &= \text{diag}(\underbrace{\lambda_{n_1}, \dots, \lambda_{n_1}}_{n_1}, \underbrace{\lambda_{n_1+n_2}, \dots, \lambda_{n_1+n_2}}_{n_2}, \dots, \underbrace{\lambda_n, \dots, \lambda_n}_{n_r}). \end{aligned}$$

By multiplying both sides of (a) by  $P^{-1}$  on the left and by  $P$  on the right, we get

$$\underbrace{[P^{-1}HP, [P^{-1}HP, \dots, [P^{-1}HP, P^{-1}NP] \dots]]}_{m\text{-fold}} = 0.$$

Observing the fact that  $[\text{diag}(a_1, a_2, \dots, a_n), E_{ij}] = (a_i - a_j)E_{ij}$  where  $E_{ij}$  is a real  $n \times n$  matrix whose  $(i, j)$ -th entry is 1 and all the other entries are 0, we see that the  $(i, j)$ -th entry of the left hand side of the equation is  $(\lambda_i - \lambda_j)^m n'_{ij}$  where  $n'_{ij}$  is the  $(i, j)$ -th entry of  $P^{-1}NP$ . It follows that  $P^{-1}NP$  is a block diagonal matrix,

$$P^{-1}NP = \text{diag}(B_1, \dots, B_r), \quad B_i \in M(n_i, C).$$

Assume that  $P_i \in GL(n_i, C)$  diagonalizes  $B_i$  respectively, namely  $P_i B_i P_i^{-1}$  is a diagonal matrix for  $i = 1, 2, \dots, r$ . Such  $P_i$ 's exist because the eigenvalues of  $B_i$ 's are

distinct since diagonal entries of  $N$  are assumed to be distinct. We can choose a permutation matrix  $Q$  so that

$$Q \operatorname{diag}(P_1, \dots, P_r) P^{-1} N P \operatorname{diag}(P_1^{-1}, \dots, P_r^{-1}) Q^{-1} = N.$$

This equation is equivalent to

$$[N, P \operatorname{diag}(P_1^{-1}, \dots, P_r^{-1}) Q^{-1}] = 0.$$

From this and the assumption that  $N$  is a diagonal matrix with distinct diagonal entries,  $P \operatorname{diag}(P_1^{-1}, \dots, P_r^{-1}) Q^{-1}$  must be a diagonal matrix,

$$D = P \operatorname{diag}(P_1^{-1}, \dots, P_r^{-1}) Q^{-1} \quad (D : \text{diagonal matrix}).$$

Now we conclude that  $P$  must have the form

$$P = D Q \operatorname{diag}(P_1, \dots, P_r), \quad P_i \in GL(n_i, C)$$

where  $Q \in GL(n, C)$  is a permutation matrix and  $D \in GL(n, C)$  is a diagonal matrix.

Then  $H$  can be expressed as

$$\begin{aligned} H &= P(P^{-1} H P) P^{-1} \\ &= D Q \operatorname{diag}(P_1, \dots, P_r) \operatorname{diag}(\underbrace{\lambda_{n_1}, \dots, \lambda_{n_1}}_{n_1}, \dots, \underbrace{\lambda_{n_r}, \dots, \lambda_{n_r}}_{n_r}) \operatorname{diag}(P_1^{-1}, \dots, P_r^{-1}) Q^{-1} D^{-1} \\ &= D Q \operatorname{diag}(\underbrace{\lambda_{n_1}, \dots, \lambda_{n_1}}_{n_1}, \dots, \underbrace{\lambda_{n_r}, \dots, \lambda_{n_r}}_{n_r}) Q^{-1} D^{-1} \end{aligned}$$

It follows that  $H$  is a diagonal matrix. □

**Theorem 2.2** *Suppose that  $N$  is a diagonal matrix with distinct diagonal entries,  $H(0)$  is symmetric and  $m$  is even. Then for the solution  $H(t)$  of (2),  $H(\pm\infty) = \lim_{t \rightarrow \pm\infty} H(t)$  exists and is a diagonal matrix.*

**Proof.** Observing the fact that  $\operatorname{tr}(A[B, C]) = \operatorname{tr}([A, B]C)$  for any real  $n \times n$  matrices  $A, B, C$ , we have

$$\begin{aligned} \frac{d}{dt} \operatorname{tr}(NH) &= \operatorname{tr}(N \underbrace{[H, [H, \dots, [H, N] \dots]]}_{m\text{-fold}}) \\ &= \operatorname{tr}(\underbrace{[N, H] [H, [H, \dots, [H, N] \dots]]}_{(m-1)\text{-fold}}) = (-1) \operatorname{tr}(\underbrace{[H, N] [H, [H, \dots, [H, N] \dots]]}_{(m-1)\text{-fold}}) \\ &= (-1) \operatorname{tr}(\underbrace{[[H, N], H] [H, [H, \dots, [H, N] \dots]]}_{(m-2)\text{-fold}}) = (-1)^2 \operatorname{tr}(\underbrace{[H, [H, N]] [H, [H, \dots, [H, N] \dots]]}_{(m-2)\text{-fold}}) \\ &= \dots = (-1)^{m/2} \operatorname{tr}(\underbrace{[H, [H, \dots, [H, N] \dots]]}_{m/2\text{-fold}} \underbrace{[H, [H, \dots, [H, N] \dots]]}_{m/2\text{-fold}}). \end{aligned}$$

Further, since  $[A, B]^t = -[A, B^t]$  where  $A$  is symmetric, we get

$$\begin{aligned} \underbrace{[H, [H, \dots, [H, N] \dots]]^t}_{m/2\text{-fold}} &= (-1) \underbrace{[H, [H, \dots, [H, N] \dots]]^t}_{m/2\text{-fold}} \\ &= \dots = (-1)^{m/2} \underbrace{[H, [H, \dots, [H, N] \dots]]}_{m/2\text{-fold}}, \end{aligned}$$

which yields

$$\frac{d}{dt} \text{tr}(NH) = \text{tr} \left( \underbrace{[H, [H, \dots, [H, N] \dots]]}_{m/2\text{-fold}} \underbrace{[H, [H, \dots, [H, N] \dots]]^t}_{m/2\text{-fold}} \right) \geq 0.$$

Here we used the fact that  $\text{tr}(AA^t) = \sum_{i,j} a_{ij}^2 \geq 0$  for any real  $n \times n$  matrix  $A$  and the equality holds if and only if  $A = 0$ . Thus  $\text{tr}(NH)$  is monotone increasing, and is bounded from both below and above because it is a continuous function on a compact set. Therefore it converges as  $t \rightarrow \pm\infty$  and its derivatives goes to zero. From the above calculation,  $\frac{d}{dt} \text{tr}(NH) = 0$  holds if and only if  $\underbrace{[H, [H, \dots, [H, N] \dots]]}_{m/2\text{-fold}} = 0$  and

from Lemma 2.1, this is satisfied if and only if  $H$  is diagonal.  $\square$

**Theorem 2.3** Suppose that  $N$  is a diagonal matrix  $\text{diag}(\mu_1, \dots, \mu_n)$  with distinct diagonal entries,  $H(0)$  is a symmetric matrix with eigenvalues  $\lambda_1, \dots, \lambda_n$  occurring with multiplicities  $n_1, \dots, n_r$  ( $\sum_{i=1}^r n_i = n$ ), that is,

$$\lambda_1 = \dots = \lambda_{n_1}, \lambda_{n_1+1} = \dots = \lambda_{n_1+n_2}, \dots, \lambda_{n_1+\dots+n_{r-1}+1} = \dots = \lambda_n,$$

and  $m$  is even. Then the dynamical system (2) has  $\frac{n!}{n_1! \dots n_r!}$  fixed points of the form  $\text{diag}(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)})$  where  $\pi$  is some permutation on  $n$  letters and the eigenvalues of the linearization of (2) at each fixed point are

$$-(\lambda_{\pi(i)} - \lambda_{\pi(j)})^{m-1} (\mu_i - \mu_j) \quad (i, j = 1, \dots, n).$$

Thus exactly one of these fixed points is asymptotically stable, where  $\mu_1, \dots, \mu_n$  and  $\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)}$  are similarly ordered.

**Proof.** From Lemma 2.1, all the fixed points of the dynamical system (2) are diagonal matrices and their diagonal entries are some permutation of the eigenvalues

of  $H(0)$  because of the isospectral property.

By using the relations  $\frac{\partial}{\partial h_{ij}} H = E_{ij}$ ,  $\partial[A, B] = [\partial A, B] + [A, \partial B]$  for any differential operator  $\partial$ , and  $[\text{diag}(a_1, a_2, \dots, a_n), E_{ij}] = (a_i - a_j)E_{ij}$ , we obtain

$$\begin{aligned}
& \left. \frac{\partial}{\partial h_{ij}} \underbrace{[H, [H, \dots, [H, N] \dots]]}_{m\text{-fold}} \right|_{H=\text{diag}(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)})} \\
&= \left\{ \underbrace{[E_{ij}, [H, \dots, [H, N] \dots]]}_{m\text{-fold}} + \underbrace{[H, [E_{ij}, \dots, [H, N] \dots]]}_{m\text{-fold}} \right. \\
&\quad \left. + \dots + \underbrace{[H, [H, \dots, [E_{ij}, N] \dots]]}_{m\text{-fold}} \right\} \Big|_{H=\text{diag}(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)})} \\
&= \underbrace{[H, [H, \dots, [E_{ij}, N] \dots]]}_{m\text{-fold}} \Big|_{H=\text{diag}(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)})} \\
&= - \underbrace{[H, [H, \dots, [N, E_{ij}] \dots]]}_{m\text{-fold}} \Big|_{H=\text{diag}(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)})} \\
&= -(\lambda_{\pi(i)} - \lambda_{\pi(j)})^{m-1} (\mu_i - \mu_j) E_{ij}.
\end{aligned}$$

This equation can be read as a characteristic equation with an eigenvalue  $-(\lambda_{\pi(i)} - \lambda_{\pi(j)})^{m-1} (\mu_i - \mu_j)$  and an eigenvector  $E_{ij}$ , showing that all the eigenvalues of the linearization of (2) at the fixed point  $H = \text{diag}(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)})$  have non-positive real parts if and only if  $\mu_1, \dots, \mu_n$  and  $\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)}$  are similarly ordered. Theorem 2.2 also guarantees that the dynamical system (2) has at least one asymptotically stable fixed point if  $m$  is even. Thus exactly one of the fixed points is asymptotically stable, where  $\mu_1, \dots, \mu_n$  and  $\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)}$  are similarly ordered.  $\square$

We can also obtain the following corollary by giving  $m$  an odd value in the calculations used in the proofs of Theorems 2.1 and 2.3.

**Corollary 2.1** *If  $m$  is an odd number,*

- (a) *the dynamical system (2) defines an isospectral flow,*
- (b) *the solution  $H(t)$  of (2) leaves  $\text{Sym}(n, R)$  even if  $H(0)$  is symmetric, and*
- (c) *the dynamical system (2) has no asymptotically stable fixed point if  $H(0)$  is diagonalizable.*



## 2.2 Extended generalization

In this section, we present an example of a little more generalized bracket equations which have different convergence properties from the equations introduced in the above section,

$$\dot{H}(t) = [H(t), [H(t), [N, [H(t), N]]]] \quad (6)$$

where  $N$  is a diagonal matrix with distinct diagonal entries and  $H(0)$  is a symmetric matrix. That (6) defines an isospectral flow on the set of symmetric matrices is proved by the same way as the proof of Theorem 2.1. By the same calculation as used in the proof of Theorem 2.3, the eigenvalues of the linearization of (6) at the diagonal point  $H = \text{diag}(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)})$  are

$$-(\lambda_{\pi(i)} - \lambda_{\pi(j)})^2 (\mu_i - \mu_j)^2 \quad (i, j = 1, \dots, n),$$

and it follows that all the diagonal points are exponentially stable fixed points if the initial matrix  $H(0)$  has distinct eigenvalues. On the other hand, it is uncertain whether or not (6) has fixed points other than diagonal points and whether the global convergence of (6) to diagonal points is guaranteed.

We can easily construct many examples of multiple bracket equations with various convergence properties by the same way.

## 3 Isospectral Gradient Flows

### 3.1 Derivation of the equation

Let  $G \subset GL(n, C)$  be a linear Lie group and  $\mathfrak{g}$  its Lie algebra. The adjoint orbit of  $G$  which passes through  $A_0 \in \mathfrak{gl}(n, C)$  is the subset of  $\mathfrak{gl}(n, C)$  defined as

$$\Omega_{A_0}^G = \{g^{-1}A_0g \mid g \in G\}.$$

Now we can say that “isospectral flows” are the flows on adjoint orbits.

In this section, we derive the equation of the gradient flow on the adjoint orbit  $\Omega_{A_0}^G$  of the real linear Lie group  $G \subset GL(n, R)$  which passes through the real matrix  $A_0 \in \mathfrak{gl}(n, R)$  of an arbitrary potential function  $\psi : \mathfrak{gl}(n, R) \rightarrow R$ .

Let the inner product of the Lie algebra  $\mathfrak{gl}(n, R)$  be defined as

$$\mu(X, Y) = \text{tr}(X^t Y).$$

Then, for arbitrary  $X, Y, Z \in \mathfrak{gl}(n, R)$ ,

$$\mu(X, [Y, Z]) = -\mu([X, Y^t], Z) \quad (7)$$

holds.

An arbitrary element  $g$  in the neighborhood of the identity element of  $G$  can be expressed as  $g = e^X$  with  $X \in \mathfrak{g}$ . Observing that

$$e^{-tX} A e^{tX} = A + t[A, X] + O(t^2)$$

for arbitrary matrices  $A$  and  $X$ , we see that an arbitrary element of the tangent space of the adjoint orbit  $\Omega_{A_0}^G$  at  $A \in \Omega_{A_0}^G$  can be expressed as  $[A, X]$  with  $X \in \mathfrak{g}$ .

Let  $\{X_k\}$  be the  $\mu$ -orthonormal basis of  $\mathfrak{g}$ . The directed derivative of  $\psi(A)$  at  $A$  along the direction  $[A, X_k]$  is

$$\sum_{i,j} \frac{\partial \psi}{\partial a_{ij}} [A, X_k]_{ij} = \mu\left(\left(\frac{\partial \psi}{\partial a_{ij}}\right), [A, X_k]\right).$$

The steepest descent direction of  $\psi(A)$  which is the negative of the linear combination of  $[A, X_k]$ 's weighted by these coefficients is calculated as follows using (7):

$$\begin{aligned} & -\sum_k \mu\left(\left(\frac{\partial \psi}{\partial a_{ij}}\right), [A, X_k]\right) [A, X_k] \\ &= -[A, \sum_k \mu\left(\left(\frac{\partial \psi}{\partial a_{ij}}\right), [A, X_k]\right) X_k] \\ &= [A, \sum_k \mu\left(\left(\frac{\partial \psi}{\partial a_{ij}}\right), A^t\right), X_k] X_k \\ &= [A, \pi_{\mathfrak{g}}\left(\left(\frac{\partial \psi}{\partial a_{ij}}\right), A^t\right)] = -[A, \pi_{\mathfrak{g}}[A^t, \left(\frac{\partial \psi}{\partial a_{ij}}\right)]], \end{aligned}$$

where  $\pi_{\mathfrak{g}}$  is the  $\mu$ -orthogonal projection from  $\mathfrak{gl}(n, R)$  to its subspace  $\mathfrak{g}$ . Then the gradient equation is

$$\frac{dA(t)}{dt} = -[A(t), \pi_{\mathfrak{g}}[A(t)^t, \left(\frac{\partial \psi}{\partial a_{ij}}\right)]]. \quad (8)$$

### 3.2 The convergence of the flow

In the case that the potential function  $\psi(A)$  is bounded below,  $A(\infty) = \lim_{t \rightarrow \infty} A(t)$  exists for the solution  $A(t)$  of (8).

The time-derivative of  $\psi(A(t))$  along the orbit  $A(t)$  is calculated as follows using (7):

$$\begin{aligned}
\frac{d}{dt}\psi(A(t)) &= \sum_{i,j} \frac{\partial\psi}{\partial a_{ij}} \frac{da_{ij}}{dt} \\
&= -\sum_{i,j} \frac{\partial\psi}{\partial a_{ij}} [A, \pi_{\mathfrak{g}}[A^t, \left(\frac{\partial\psi}{\partial a_{ij}}\right)]]_{ij} \\
&= -\mu\left(\left(\frac{\partial\psi}{\partial a_{ij}}\right), [A, \pi_{\mathfrak{g}}[A^t, \left(\frac{\partial\psi}{\partial a_{ij}}\right)]]\right) \\
&= \mu\left(\left(\frac{\partial\psi}{\partial a_{ij}}\right), A^t, \pi_{\mathfrak{g}}[A^t, \left(\frac{\partial\psi}{\partial a_{ij}}\right)]\right) \\
&= -\mu([A^t, \left(\frac{\partial\psi}{\partial a_{ij}}\right)], \pi_{\mathfrak{g}}[A^t, \left(\frac{\partial\psi}{\partial a_{ij}}\right)]).
\end{aligned}$$

Because  $\dot{\psi} \rightarrow 0(t \rightarrow \infty)$ , we conclude that

$$\pi_{\mathfrak{g}}[A(\infty)^t, \left(\frac{\partial\psi}{\partial a_{ij}}\right)] = 0 \quad (9)$$

is satisfied at  $A(\infty)$ .

### 3.3 Flows on the set of complex matrices

In this section, the discussions in the above sections are generalized for the adjoint orbit  $\Omega_{A_0}^G$  of  $G \subset GL(n, C)$  which passes through  $A_0 \in \mathfrak{gl}(n, C)$  and a potential function  $\psi : \mathfrak{gl}(n, C) \rightarrow R$ . Note that we can't use complex differentiation for the potential function  $\psi : \mathfrak{gl}(n, C) \rightarrow R$  since it is singular (excepting when  $\psi$  is a constant function). In the derivation of the gradient equation, we treat  $\mathfrak{gl}(n, C)$  not as the  $n^2$ -dimensional linear space on  $C$ , but as the  $2n^2$ -dimensional linear space on  $R$ .

We state the results without proof. Let  $a_{ij}$  and  $b_{ij}$  be the real and imaginary part of the  $(i, j)$ -th entry of  $A$  respectively. We define the inner product of  $\mathfrak{gl}(n, C)$  as

$$\mu(X + \tilde{X}i, Y + \tilde{Y}i) = \text{tr}(X^t Y + \tilde{X}^t \tilde{Y}),$$

where  $X, \tilde{X}, Y, \tilde{Y}$  are real matrices. Then the gradient equation is

$$\frac{dA(t)}{dt} = -[A(t), \pi_{\mathfrak{g}}[A(t)^*, \left(\frac{\partial\psi}{\partial a_{ij}} + \frac{\partial\psi}{\partial b_{ij}}i\right)]] \quad (10)$$

and

$$\pi_{\mathfrak{g}}[A(\infty)^*, \left(\frac{\partial\psi}{\partial a_{ij}} + \frac{\partial\psi}{\partial b_{ij}}i\right)] = 0 \quad (11)$$

is satisfied at  $A(\infty)$ .

### 3.4 Previous examples

In this section, we derive two already known isospectral flows as special cases of (8) and (10).

#### 1. Brockett[1]'s flow.

Let  $G = SO(n)$ ,  $\psi(A) = -\text{tr}(AN)$  where  $N$  is a real constant symmetric matrix and  $A(0)$  be a real symmetric matrix in (8). Then we get the Brockett's equation (1)

$$\dot{A}(t) = [A(t), [A(t), N]]. \quad (12)$$

Note that we can omit  $\pi_{\mathfrak{so}(n)}$  in this equation because  $[A(t), N] \in \mathfrak{so}(n)$  where  $A(t)$  and  $N$  are symmetric.

#### 2. The dynamical system that diagonalizes skew-Hermitian matrices.

Let  $G = SU(n)$ ,  $\psi(A) = \frac{1}{2} \sum_{i \neq j} a_{ij}^2 + b_{ij}^2$  and  $A(0) \in \mathfrak{su}(n)$  in (10). Then we get the dynamical system on  $\mathfrak{su}(n)$

$$\dot{A}(t) = -[A(t), [A(t), J(A(t))]], \quad (13)$$

where  $J(A)$  is the diagonal matrix obtained by replacing all the non-diagonal entries of  $A$  by zeros. Note that we can omit  $\pi_{\mathfrak{su}(n)}$  in this equation because  $[A(t), J(A(t))] \in \mathfrak{su}(n)$  where  $A(t) \in \mathfrak{su}(n)$ . It was shown that the solution  $A(t)$  of this dynamical system converges to a diagonal matrix for almost all initial values  $A(0) \in \mathfrak{su}(n)$  in Nakamura[8].

### 3.5 A new dynamical system for the eigenvalue problem of non-symmetric matrices

In this section, we derive a new dynamical system that non-symmetric complex matrices as a special case of (10).

Let  $G = GL(n, C)$ ,  $\psi(A) = \frac{1}{2} \sum_{i > j} a_{ij}^2 + b_{ij}^2$  and  $A(0) \in \mathfrak{gl}(n, C)$  in (10). Then we get the dynamical system on  $\mathfrak{gl}(n, C)$

$$\dot{A}(t) = -[A(t), [A(t)^*, L(A(t))]], \quad (14)$$

where  $L(A)$  is the matrix obtained by replacing all the upper triangular entries of  $A$  by zeros.

It is proved in the following Lemma 3.1 that the fixed point condition (11) for this dynamical system, namely,  $[A(t)^*, L(A(t))] = 0$  is satisfied if and only if  $A(t)$  is an

upper triangular matrix. Then the solution  $A(t)$  of the dynamical system converges to an upper triangular matrix for an arbitrary initial value  $A(0)$ . This dynamical system provides a new computational algorithm for solving the eigenvalue problem of non-symmetric complex matrices.

**Lemma 3.1** *For any complex matrix  $A$ , the following two statements are equivalent.*

- (a)  $[A^*, L(A)] = 0$ .
- (b)  $A$  is an upper triangular matrix.

**Proof.** (b) $\Rightarrow$ (a) is obvious. To prove (a) $\Rightarrow$ (b) by induction, it is enough to show the following two statements.

- (i)  $a_{i1} = 0 \quad (2 \leq i \leq n)$ .
- (ii) For  $k = 2, 3, \dots, n-1$ ,

$$\begin{aligned} a_{i1} &= 0 \quad (2 \leq i \leq n), \\ a_{i2} &= 0 \quad (3 \leq i \leq n), \\ &\vdots \\ a_{i,k-1} &= 0 \quad (k \leq i \leq n) \\ &\Downarrow \\ a_{ik} &= 0 \quad (k+1 \leq i \leq n). \end{aligned}$$

The (1,1)-th entry of  $[A^*, L(A)]$  is calculated as follows,

$$[A^*, L(A)]_{11} = \sum_{2 \leq i \leq n} \overline{a_{i1}} a_{i1}.$$

This is enough to show (i). Under the assumptions of ii), the  $(k,k)$ -th entry of  $[A^*, L(A)]$  is calculated as follows,

$$[A^*, L(A)]_{kk} = \sum_{k+1 \leq i \leq n} \overline{a_{ik}} a_{ik}$$

This is enough to show (ii). □

## 4 Concluding Remarks

Brockett[2] already derived results very similar to our results stated in section 3 of this paper using the Killing metric  $\kappa$  instead of the metric  $\mu$ . His results is described in the following theorem.

**Theorem 4.1** *Let  $G$  be a real compact semi-simple Lie group and let  $\mathfrak{g}$  be its Lie algebra. Let  $g_0 \in \mathfrak{g}$  and let  $\theta(g_0)$  be an adjoint orbit. If  $\psi : \theta(g_0) \rightarrow \mathbb{R}$  is a differentiable function then the corresponding gradient flow on  $\theta(g_0)$  is given by*

$$\dot{x} = -[x, [x, \psi_x]]. \quad (15)$$

*Along this flow  $\dot{\psi} = -\langle [x, \psi_x], [x, \psi_x] \rangle_\kappa$ . As  $t$  goes to infinity  $\psi$  approaches a constant and  $x$  approaches an equilibrium point.*

See Brockett[2] for detail. Note that Brockett[1]'s equation (12) and (13) can be derived from both our formula (10) and Brockett[2]'s formula (15), but our new dynamical system (14) can not be derived from Brockett[2]'s formula (15) since  $\mathfrak{gl}(n, \mathbb{C})$  is not compact or semi-simple. Our results can be regarded as non-compact generalization of Brockett[2]'s results.

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